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MORE ON COMPLEMENTARY TREES *

Daniel J. KLEITMAN

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

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It is shown that a graph with no multiple edges on n vertices, $n \geqslant 5$, with $2(n-2)$ arcs labelled $1, \ldots, n-1$ and $1', \ldots, n-1'$ having at least one spanning tree whose arcs include no pair (j, j'), has at least six of them. This is a result of Rothblum. It is also shown that if the graph has only one multiple edge and that is a double edge, and $n \geqslant 4$, then it has at least four such spanning trees.

1. Introduction

A number of authors have studied "complementary trees in graphs". Given a graph G on (without loops or multiple arcs) n vertices having $2n-2$ arcs labelled $1, \ldots, n-1$ and $1', \ldots, n-1'$ such that arcs $1, \ldots, n-1$ form a tree. Danzig [2] showed that there must exist a second tree having no two arcs j and j' both in it. Adler [1] showed further that a total of at least four such trees must exist in this circumstance. The present author using a different technique (also described by Chvátal), proved Adler's theorem and some related results, including the following two remarks.

Remark 1. A graph as above with exactly one doubled edge and no loops for $n \geqslant 3$ must have at least three such trees.

Remark 2. A graph as above with no loops must have at least two such trees.

Magnanti [6] considered similar problems in the more general context of matroids, while Greene et al. [4] obtained still more related results

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along with an upper bound in the number of such trees in such a graph.

Rothblum [8] showed that for $n \geq 5$, at least six of these trees must exist and independently obtained the same upper bound.

It is the purpose of this note to provide a more concise proof of Rothblum's result, along with some related results (e.g., that if $n \geq 4$ the three trees of Remark 1 can be replaced by four trees) and further conjectures.

2. Results

We will prove the following theorem.

Theorem. *A graph on n vertices, $n \geq 5$, having no multiple edges or loops, with $2(n-1)$ arcs one labelled each of i or i' for $1 \leq i \leq n-1$, for which the arcs $1, \ldots, n-1$ form a tree, has at least six trees in which all $n-1$ indices appear.*

Such trees will be called good trees below. All graphs mentioned below will be considered to have $2n-2$ arcs labelled as above and no loops.

The argument proceeds through a number of remarks or lemmas some of which have independent interest. We first show that any arc in a minimal sized graph G_s violating the theorem must have at least one good tree that contains it, and hence by Remark 1 above at least two such trees.

We next show that a graph containing no loops and only one doubled edge must have at least four good trees if it has one for $n \geq 4$. This implies that the minimum degree of a vertex in G_s must be three, because the good trees containing either arc incident to this vertex are trees on a contracted graph containing only one doubled edge.

A similar but more complicated result holds for graphs containing two doubled edges. Since all graphs on n vertices with $2n-2$ arcs have vertices of degree ≤ 3 and trees containing arcs incident to such vertices are trees on a contracted graph having at most two doubled edges, these results can be used to delimit the properties of graphs having few trees. In fact they imply that no graph G_s can have only four trees, and with one additional short argument imply that there can be no G_s at all.

3. Argument

We call arcs j' and j alternates below and call a set of arcs having no two arcs with the same number (i.e., no k, k') a good set of arcs.

Remark A. (Rado [7]). *If in a graph G arc j appears in no good tree while at least one such tree exists, then there is a good set of arcs that span j and their own alternates.*

Proof. Choose a good tree and remove j' from it; if the remaining arcs span $G - \{j'\}$ they form the desired set. If they fail to span arc k, remove k from G and k' from the remaining tree arcs. If an arc is not spanned, again remove it and its alternate; continue the process until the resulting arc sets spans its complement. If j is ever not spanned, we can form a new good tree containing j by switching arcs, replacing the arc whose removal left j unspanned by j, the arc whose removal left that arc's alternate unspanned by its alternate, etc. and we will have a good tree containing j.

We call a graph *reducible* if it has a good tree, and all such trees contain a certain arc.

Remark B. *The graph(s) G_5 (having no multiple edges) on fewest vertices $\geqslant 5$ having at least one but five or fewer good trees, is not reducible.*

Proof. Given a reducible graph G we can apply Remark A (if necessary more than once) to find a subgraph, which by suitable identification of vertices can be connected, that possesses a good tree, is not reducible and has an extra arc in it, with an extraneous label.

It cannot have four vertices since it could then not have the required seven arcs without a multiple edge, and likewise cannot have three vertices or less. Since a tree on the subgraph can be extended to one in G, the subgraph must also have five or fewer good trees.

Remark C. *If a graph G has only one multiple edge, that edge double, and has four or more vertices, it has four good trees if one.*

Proof. As in the previous argument, if G is irreducible we can apply Remark A (and perhaps identification of vertices) to reduce G to a smaller graph that is irreducible, has at most one multiple arc and an extraneous arc. This graph must have four or more vertices, hence has an index j that is possessed by neither multiple arc. By Remark 2 above there must be two trees containing j and also j'.

Remark D. *If a graph G for $n \geq 4$ has two pairs of multiple edges both double and these edges are adjacent and either the arc joining their other vertices is not in G, or three indices occur among the four multiple arcs, then G has three or more good trees if one. If all four indices in the multiple arcs are distinct (no pair of alternates) then G has four or more good trees if one.*

Proof. As in the previous case we can obtain an irreducible subgraph. If that subgraph has four vertices, some arc in it will be non-multiple and at least two trees must contain it so that there will be three trees. The only case in which we can have only two trees is if there are only three vertices in the irreducible subgraph, and two indices among the four multiple arcs. This can only occur if the extraneous arc was the other arc spanned by the multiple arcs, and the arc indices are two pairs of alternates.

If all four multiple arc indices are distinct, there could be only three trees only if the irreducible subgraph has $n = 5$, since there would be four trees if it had only one multiple arc or less, or had an index not in a multiple arc. If $n = 5$, if any non-multiple arc is not in a triangle, or is only in a triangle one side of which is a multiple edge containing its alternate, there will be at least four trees, since at least three trees will contain that arc by the $n = 4$ discussion. The only irreducible graphs with $n = 5$ left to consider have two arc disjoint triangles each based on a multiple edge, each having all four indices in it; each such graph can be seen to contain at least four good trees.

Remark E. *No G with five or more vertices and no multiple arcs has four trees. If G_s has five trees each degree 3 vertex must form a corner of a tetrahedron in G_s whose vertices have two duplicated indices. The arcs in G_s corresponding to each pair of alternates in the tetrahedron are not adjacent. G_s can have no degree two vertices.*

Proof. Let G be irreducible. It has average degree less than four. If it had a degree two vertex v there would be at least four trees containing either arc incident to v by Remark C, since trees on G containing an arc are trees on the graph obtained from G by contracting on that arc and omitting its alternate, and contractions on such an arc would produce at most one multiple arc. G_s can have no degree one vertex. It must, therefore, possess degree three vertices. By Remarks D and C four trees will contain an arc incident to one of these unless it is a corner of a tetrahedron and unless the four doubled arcs obtained by contracting it include a repeated index, and not its own index. If this is to occur for all three arcs incident to the vertex, either, there are three repeated arc indices in the tetrahedron, or two are repeated as indicated in the statement of this remark. In the former case there are four ways to arrange trees in the tetrahedron and if the tetrahedron is contracted to a point, at least two ways of arranging trees in the rest of G_s; yielding at least eight good trees altogether.

Proof of the Theorem. By Remark E, G_s must contain a degree three vertex v with arcs (v, v_1), (v, v_2), (v, v_3), (v_1, v_2), (v_2, v_3) and (v_1, v_3) that without loss of generality can be assigned respective indices 1, 2, 3, 3′, 1′, 4. There are five good trees in G_s. By Remark E, at least three trees contain 1. Further the trees containing 1 vary only in indices 2, 3 and 4; for if they varied in some other index j, there would be four of them by Remark 2 applied to the graphs obtained by contracting arcs j and j'. That means that for any other index j, the three trees containing 1 contain j (say) and the remaining two trees (containing 1′) contain j'. By the same argument applied to 3 instead of 1, the three good trees containing j must be those that contain 3. This is a contradiction unless there are only four indices, so that $n = 5$, in which case there must be a degree two vertex in G_s; which cannot be, again by Remark E.

4. Conjectures

The $n = 5$ graph considered in the last argument in fact has exactly six good trees. Moreover, it is obviously possible to construct reducible graphs that reduce to it for any larger n, which again possess exactly six good trees. It is not so easy to construct an irreducible graph with these same properties. In fact one might conjecture that there are no irreducible graphs on $n > 5$ vertices with only six good trees and that the minimum number of good trees in any irreducible graph grows unboundedly with n.

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